ITEP-TH-13/00
hep-th/0003055

March 2000

Wilson Loops in $\mathcal{N} = 4$ Supersymmetric Yang–Mills Theory

J.K. Erickson[1], G.W. Semenoff[3] and K. Zarembo[1,2,4]

Email: erickson@nbi.dk, semenoff@ias.edu, zarembo@physics.ubc.ca

[1] *Department of Physics and Astronomy*
[2] *Pacific Institute for Mathematical Sciences*
University of British Columbia
Vancouver, British Columbia, Canada V6T 1Z1

[3] *Institute for Advanced Study,*
Einstein Drive, Princeton, New Jersey 08540 USA

[4] *Institute of Theoretical and Experimental Physics*
B. Cheremushkinskaya 25, Moscow, 117218 Russian Federation

Abstract

Perturbative computations of the expectation value of the Wilson loop in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory are reported. For the two special cases of a circular loop and a pair of anti-parallel lines, it is shown that the sum of an infinite class of ladder-like planar diagrams, when extrapolated to strong coupling, produces an expectation value characteristic of the results of the AdS/CFT correspondence, $\langle W \rangle \sim \exp\bigl((\text{constant})\sqrt{g^2N}\bigr)$. For the case of the circular loop, the sum is obtained analytically for all values of the coupling. In this case, the constant factor in front of $\sqrt{g^2N}$ also agrees with the supergravity results. We speculate that the sum of diagrams without internal vertices is exact and support this conjecture by showing that the leading corrections to the ladder diagrams cancel identically in four dimensions. We also show that, for arbitrary smooth loops, the ultraviolet divergences cancel to order g^4N^2.

1 Introduction and Summary

The AdS/CFT correspondence conjectures a duality between $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions and type IIA superstring theory on an $\mathrm{AdS}_5 \times S^5$ background [1, 2, 3, 4, 5]. There are three levels to this conjecture:

- In its strongest version the correspondence asserts that there is an exact equivalence between four dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and type IIB superstring theory on the $\mathrm{AdS}_5 \times S^5$ background. This also contains the conjecture that the $\mathrm{AdS}_5 \times S^5$ background is an exact solution of type IIB superstring theory.

- A weaker version asserts a duality of the 't Hooft limit of the gauge theory, where $N \to \infty$ with the 't Hooft coupling g^2N held fixed, and the classical $g_s \to 0$ limit of type IIA superstring theory on $\mathrm{AdS}_5 \times S^5$. In this correspondence, corrections to classical supergravity theory from stringy effects which are of order α' would agree with corrections to the large 't Hooft coupling limit, of order $1/\sqrt{g^2N}$, but higher orders in g_s on the supergravity side and non-planar diagrams on the gauge theory side could disagree.

- An even weaker version is a duality between the 't Hooft limit where one also takes the strong coupling limit $g^2N \to \infty$ and the low energy, supergravity limit of type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$. In this case, there would be order α' and g_s corrections to supergravity which might not agree with order $1/N^2$ and $1/\sqrt{g^2N}$ corrections to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory.

Even this last, weakest version of the correspondence has profound consequences. Previous to it, the only quantitative tool which could be used to attack supersymmetric Yang-Mills theory was perturbation theory in g^2, the Yang-Mills coupling constant. This is limited to the regime where g^2 is small. Furthermore, although some qualitative features of the large N limit are known, it is not possible to sum planar diagrams explicitly. The AdS/CFT correspondence enables one to compute correlation functions in the large N, large g^2N limit. This limit contains the highly nontrivial sum of all planar Feynman diagrams, and emphasizes those diagrams which have infinitely many vertices.

Because they are inaccessible to perturbation theory, the predictions of the AdS/CFT correspondence are very difficult to check in any direct way. The main evidence which supports the correspondence comes from symmetry arguments. The global symmetries of both $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and type IIB string theory on

$\mathrm{AdS}_5 \times S^5$ are identical. They have the same global super-conformal group $SU(2,2 \mid 4)$ (whose bosonic subgroup is $SO(4,2) \times SU(4)$). Not only are the global symmetries the same, but some of those objects which carry the representations of the symmetry group—the spectrum of chiral operators in the field theory and the fields in supergravity theory—can, to some degree, be matched [3]. Furthermore, both theories are conjectured to have an Montonen-Olive $SL(2,Z)$ duality acting on their coupling constants.

The only correlation functions which can be checked directly are correlators which are protected by non-renormalization theorems and thus do not depend on the coupling constant. These correlators are related to anomalies in the $SU(4)$ R-symmetry and conformal current algebras. A number of them have been shown to match [4].

In this paper, we sum infinite classes of planar ladder diagrams in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, and compare our results with the predictions of the AdS/CFT correspondence. We are for the most part interested in the Wilson loop operator. All computations are in the 't Hooft limit where N is put to infinity while holding g^2N fixed. We shall not succeed in providing a direct test of the correspondence. However, we will show that the generic dependence of the loop expectation value on the coupling constant found in the AdS/CFT computations arises naturally when perturbation theory is summed to all orders and extrapolated to strong coupling. We also compute the exact weak coupling results to order g^4N^2 which, if one takes the strongest version of the AdS/CFT correspondence seriously, should tell us something about classical superstrings on $\mathrm{AdS}_5 \times S^5$ in the strong curvature limit. We shall also find that leading order corrections to the sum of ladder diagrams vanish for the two simple loops that we study.

1.1 The Wilson Loop

The Wilson loop operator is a phase factor associated with the trajectory of a heavy quark in the fundamental representation of the gauge group, which in our case is $U(N)$. The loop operator which couples to classical quantities in the superstring theory in the simplest way provides a source for a classical string [6, 7, 8, 9]. It is

$$W(C) = \frac{1}{N} \,\mathrm{Tr}\, \mathcal{P} \exp \oint_C d\tau \left(iA_\mu(x)\dot{x}_\mu + \Phi_i(x) y_i \right), \tag{1}$$

where $x_\mu(\tau)$ is a parameterization of the loop, $y_i = \sqrt{\dot{x}^2}\theta_i$ and θ_i is a point on the five dimensional unit sphere ($\theta^2 = 1$). This operator measures the holonomy of a heavy W-boson whose mass results from spontaneous breaking of $U(N+1)$ gauge symmetry

to $U(N) \times U(1)$. In the AdS/CFT correspondence, it is computed by finding the area of the world-sheet of the classical string in $\mathrm{AdS}_5 \times S^5$ whose boundary is the loop C, which in turn lies on the boundary of AdS_5 [6, 7, 4].

The expectation value of the Wilson loop operator is easy to compute in perturbation theory

$$\langle W(C) \rangle = 1 + \frac{g^2 N}{4\pi^2} \oint_C d\tau_1\, d\tau_2 \frac{|\dot{x}(\tau_1)||\dot{x}(\tau_2)| - \dot{x}(\tau_1) \cdot \dot{x}(\tau_2)}{|x(\tau_1) - x(\tau_2)|^2} + \cdots . \tag{2}$$

For a loop without cusps or self-intersections, this result is finite. This is because a cancellation occurs between the contributions of the scalar and vector fields. The case of cusps and self-intersections has been discussed in [8].

1.2 Summary of results

We work in Feynman gauge and consider specific, not necessarily gauge invariant, classes of planar Feynman diagrams. We will comment on the issue of gauge invariance later. In all computations, we consider only planar diagrams, thus the coupling constant dependence of all results is in the combination $g^2 N$.

1.2.1 Cancellation of ultraviolet divergences

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory is a conformal field theory for any value of its coupling constant. This conformal invariance is a result of the fact that the theory has no dimensional coupling constants, so it is conformally invariant at the tree level. In addition, the high degree of supersymmetry leads to a cancellation of loop corrections to the coupling constant renormalization.

However, as we shall see, the theory does have divergent wave-function renormalization, which does not contradict ultraviolet finiteness, since all divergences can be absorbed by rescaling the fields[1]. The Wilson loop potentially contains its own short-distance singularities which occur when the spacetime arguments of operators in the loop approach each other. In (2), the divergent contributions of the gauge fields and the scalars mutually cancel. We shall see in the following that, to order $g^4 N^2$, part of such singularities survive in order to compensate for the infinite wave function renormalization, and all ultraviolet singularities cancel for a generic smooth (without cusps and self-intersections) loop.

[1] In fact, some divergences are necessary in order for the theory to solve the conformal bootstrap equations [10].

1.2.2 Parallel lines

The Wilson loop operator for a single infinite straight line, $W(C)$ where C is parameterized by $x(s) = (s, 0, 0, 0)$, is a BPS object: it commutes with half of the sixteen supercharges [8]. This residual supersymmetry protects it from obtaining quantum corrections. It is easy to verify that the first few orders in weak coupling perturbation theory cancel. In the strong coupling limit of planar diagrams, the AdS/CFT correspondence predicts that

$$\langle W(C) \rangle = 1. \tag{3}$$

In fact, this is the case for any array of parallel straight lines with like orientations.

1.2.3 Circular loop

For the circular loop, conformal invariance predicts that the expectation value of the loop operator is independent of the radius of the loop[2]. We find that the sum of all planar Feynman diagrams which have no internal vertices (which includes both rainbow and ladder diagrams) produces the expression

$$\langle W(C) \rangle_{\text{ladders}} = \frac{2}{\sqrt{g^2 N}}\, I_1(\sqrt{g^2 N}), \tag{4}$$

where I_1 is the Bessel function. Taking the large $g^2 N$ limit gives

$$\langle W(C) \rangle_{\text{ladders}} = \frac{\mathrm{e}^{\sqrt{g^2 N}}}{(\pi/2)^{1/2} (g^2 N)^{3/4}}, \tag{5}$$

which has exponential behavior identical to the prediction of the AdS/CFT correspondence [8, 11],

$$\langle W(C) \rangle_{\text{AdS/CFT}} = \mathrm{e}^{\sqrt{g^2 N}}. \tag{6}$$

It is intriguing that this sum of a special class of diagrams produces the exact asymptotic behavior that is predicted by the AdS/CFT correspondence, considering that it does not include any diagrams which have internal vertices. Assuming that the AdS/CFT prediction is indeed the correct asymptotic behavior, there are two

[2]Note the subtle fact that, since the circle could be obtained from an infinite straight line by a conformal transformation, and this conformal transformation should be a symmetry of the vacuum, the circular loop should have expectation value equal to that of the infinite straight line (3). We shall see explicitly that this is not the case. This asymmetry is introduced by the boundary conditions which require that propagators go to zero at infinity.

possible reasons for this agreement. First, the corrections to the sum of planar ladder diagrams could produce a term which would be added to (4) and which would grow no faster than $\exp(\sqrt{g^2N})$ for large g^2N. Such a term could modify the prefactor of the exponential but would not modify the exponent. Second, it is possible that corrections to the sum of ladder diagrams cancel and the result (4) is exact.

In order to explore this appealing possibility, we compute the leading order corrections to (4) coming from diagrams with internal vertices. These occur at order g^4N^2. We find that these diagrams do indeed cancel exactly when the spacetime dimension is four. Away from dimension four, there is a residual term of order $(D-4)g^4N^2$. It is tempting to speculate that all higher order corrections from diagrams with internal vertices also cancel. At this point we have not been able to check this possibility beyond order g^4N^2. One reason we have to be optimistic is that the circular loop is related to the single straight line by a conformal transformation. The conformal transformation must not be representable as a unitary operator operating on the loop operator. Otherwise, the expectation value of the circular loop would be one (identical to the straight line) and we know this is not the case. However, for some classes of diagrams, the symmetry could survive. We know that, for the straight line, the rainbow diagrams, which are what the ladder diagrams are mapped onto, cancel identically. This clearly does not happen for their conformal images on the circle. However it could still happen for the diagrams which have internal vertices—which do in fact cancel identically for the straight line Wilson loop. As stated above, we have checked explicitly that this is indeed the case for the leading order g^4N^2 corrections. This has not yet been checked at higher orders.

One important issue is gauge invariance. The computation of (4) is in Feynman gauge. Since only a subset of the diagrams has been included, there is no guarantee that the calculation is gauge invariant. In fact, summing diagrams without internal vertices in a different gauge would give a different result. However, if the corrections indeed vanish then the sum would be equal to the gauge invariant sum of all Feynman diagrams on the Wilson loop. The cancellation of the diagrams with internal vertices would then be a special property of the Feynman gauge[3]. We have explicitly demonstrated that corrections vanish up to order g^4N^2.

[3]In Feynman gauge, the vector propagator coincides with the scalar one, which makes, for instance, cancellation of ultraviolet divergences more transparent than in other gauges: the second term in (2) vanishes even before integration along the loop.

1.2.4 Anti-parallel lines

Consider a rectangular Wilson loop with length T and width L. In the limit that $T \gg L$, the ends can be ignored and the loop can be seen as a pair of anti-parallel lines separated by a distance L. We find that the sum of all planar diagrams without internal vertices, the ladder diagrams, is given by

$$\langle W(C)\rangle_{\rm ladders} = \exp\left[\left(\frac{g^2N}{4\pi} - \frac{g^4N^2}{8\pi^3}\ln\frac{1}{g^2N} + \cdots\right)\frac{T}{L}\right], \qquad g^2N \ll 1; \tag{7}$$

and by

$$\langle W(C)\rangle_{\rm ladders} = \exp\left[\left(\frac{\sqrt{g^2N}}{\pi} - 1 + \mathcal{O}\Big(\frac{1}{\sqrt{g^2N}}\Big)\right)\frac{T}{L}\right], \qquad g^2N \gg 1. \tag{8}$$

Here, we use the fact that $T \gg L$ to ignore any terms in the exponent which are of lower order than T/L.

The logarithm in the exponent of the weak coupling limit (7) comes from an infrared divergence which appears in the order g^4N^2 ladder diagram. It was shown in [12] that resummation of these logarithms to all orders replaces the logarithm of T/L by a logarithm of the coupling constant. We will elaborate on those arguments in section 5. Logarithms of this kind are known to appear in the Wilson loop in QCD at order g^6 [13]. In the weak coupling limit, the leading term and the coefficient of the next-to-leading term are indeed gauge invariant. However, the strong coupling limit depends on the gauge parameter and we have quoted the result (8) in Feynman gauge only (while the particular constants vary, the generic behavior is still $e^{\sqrt{g^2N}}$).

In the large 't Hooft coupling limit, the AdS/CFT computation[4] [6, 7]

$$\langle W(C)\rangle_{\rm AdS/CFT} = \exp\Big(\frac{4\pi^2\sqrt{g^2N}}{\Gamma^4(1/4)}\frac{T}{L}\Big), \qquad g^2N \gg 1. \tag{9}$$

Comparing this with (8), we see that summing only ladder diagrams does not produce (9). The failure of this result to agree with the supergravity computation is not very interesting: since the anti-parallel lines are not as closely related to a BPS object as a circle, we have no reason to speculate that the sum of ladder diagrams would reproduce the exact strong-coupling behavior.

We also find that the diagrams with internal vertices cancel exactly to the leading order g^4N^2. In this case, as opposed to the circular loop where they only cancel in four dimensions, the cancellation occurs in any number of spacetime dimensions.

[4]Note that the g^2 in [6] differs by a factor of 2 from our definition.

The sum over an infinite number of diagrams that we have done produces the strong coupling behavior $\exp\left[(\text{constant})\sqrt{g^2N}\right]$ which is characteristic of the Wilson loops computed using the AdS/CFT correspondence. This seems to be the generic behavior of infinite sums of planar ladder diagrams.

1.3 Outline

In the rest of this paper we give a detailed derivation of the above results. In section 2, we give the results of some loop calculations. In section 3, we prove that ultraviolet singularities cancel to order g^4N^2 for any smooth loop. In section 4, we consider the circular loop. We compute the sum over ladder diagrams and show that the diagrams with internal vertices cancel to order g^4N^2. In section 5, we give similar arguments for two anti-parallel lines. In section 6, we summarize some speculations about the implications of our results. In appendix A, we summarize our notation and conventions and record some useful formulae.

2 Perturbation theory

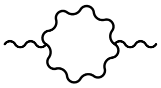
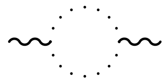
With the exception of section 3, we use regularization by dimensional reduction throughout this paper. This procedure considers supersymmetric Yang-Mills theory in 2ω dimensions as a dimensional reduction of $\mathcal{N}=1$ supersymmetric Yang-Mills theory in ten dimensions. In this scheme, the gauge field $A^a_\mu(x)$ is a 2ω component vector field. The index of the scalar field runs over $10-2\omega$ values, $i=1,...,10-2\omega$. In every dimension, the fermion field has sixteen real components. Regularization by dimensional reduction preserves the sixteen supersymmetries of the ten dimensional Yang-Mills theory. These lead to four conserved four component Majorana spinor supercharges in four dimensions. This regularization scheme provides a supersymmetric regularization of the four dimensional theory. Since the gauge coupling becomes dimensional in any spacetime dimension other than four, the regularization breaks conformal symmetry explicitly. It also modifies the R-symmetry.

2.1 One loop self energy of the vector and scalar fields

Consider the one loop self-energies of the vector and scalar field. The dimension of space-time is $D=2\omega$. All of the loop integrals are elementary and can be found in reference books such as [14]. We parameterize the Wilson loop by $x(\tau)$, and abbreviate $x^{(i)}=x(\tau^i)$.

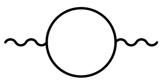
The vector field obtains self-energy corrections from:

- N^2 colors of vector fields and ghost fields:

 + 

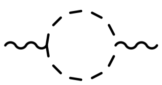
$$= \delta^{ab} g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} \cdot 2(3\omega - 1) \frac{\delta_{\mu\nu} - p_\mu p_\nu / p^2}{p^{2-2\omega}}$$

- $10 - 2\omega$ real scalar fields in the adjoint representation:



$$= -\delta^{ab} g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} \cdot (10 - 2\omega) \frac{\delta_{\mu\nu} - p_\mu p_\nu / p^2}{p^{2-2\omega}}$$

- four flavors of four-component Majorana fermions in the adjoint representation:



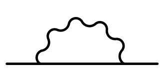
$$= -\delta^{ab} g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} \cdot 16(\omega - 1) \frac{\delta_{\mu\nu} - p_\mu p_\nu / p^2}{p^{2-2\omega}}$$

Note that these are the negative of the conventionally defined self-energies. Thus, to one loop order, the propagator for the unrenormalized gluon, in Feynman gauge is

$$\Delta^{ab}_{\mu\nu} = g^2 \delta^{ab} \frac{\delta_{\mu\nu}}{p^2} - g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} \cdot 4(2\omega - 1) \delta^{ab} \frac{\delta_{\mu\nu} - p_\mu p_\nu / p^2}{p^{6-2\omega}}. \qquad (10)$$

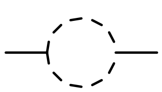
Similarly, we can compute the one loop correction to the scalar propagator. It obtains corrections from:

- the scalar-vector intermediate state:



$$= \delta^{ab} g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} \cdot 4(2\omega - 1) \frac{\delta_{ij}}{p^{6-2\omega}}$$

- and the fermion loop:



$$= -\delta^{ab} g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} \cdot 8(2\omega - 1) \frac{\delta_{ij}}{p^{6-2\omega}}$$

Thus, to one loop order, the (unrenormalized) scalar propagator is

$$D_{ij}^{ab} = g^2 \delta^{ab} \frac{\delta_{ij}}{p^2} - g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} \cdot 4(2\omega-1) \frac{\delta_{ij}\delta^{ab}}{p^{6-2\omega}}. \tag{11}$$

Note that, aside from vector indices, the scalar and vector propagators are identical. Also, note that the self-energy corrections have poles at $\omega = 2$ which arise from an ultraviolet divergence. If we were to compute correlators of local renormalized fields, it would be necessary to add a counterterm to the action in order to cancel these ultraviolet singularities. Here, for purpose of computing the Wilson loop, we leave them unrenormalized.

2.2 Ladder diagrams

The ladder-like diagrams to order $g^4 N^2$ contribute

$$\Sigma_1 = \frac{g^4 N^2}{6} \oint_{\tau_1 > \tau_2 > \tau_3 > \tau_4} d\tau_1\, d\tau_2\, d\tau_3\, d\tau_4\, \frac{(|\dot{x}^{(1)}||\dot{x}^{(2)}| - \dot{x}^{(1)} \cdot \dot{x}^{(2)})(|\dot{x}^{(3)}||\dot{x}^{(4)}| - \dot{x}^{(3)} \cdot \dot{x}^{(4)})}{(|x^{(1)} - x^{(2)}|^2 |x^{(3)} - x^{(4)}|^2)^{\omega-1}}.$$

This result is finite when $x(\tau)$ is a smooth curve.

2.3 Insertion of one loop corrections to propagators

Using the one loop self-energies of the vector and scalar fields that we found in section 2.2, we can find the corrections to the expression (2) resulting from insertion of one loop into the vector and scalar field propagators.

We begin with the correction to the scalar propagator in momentum space. From equation (11) it is

$$-\delta^{ab} g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} 4(2\omega-1) \frac{1}{[p^2]^{3-\omega}}.$$

The Fourier transform of this expression, computed using (45), is

$$-\delta^{ab} g^4 N \frac{\Gamma^2(\omega-1)}{2^5 \pi^{2\omega} (2-\omega)(2\omega-3)} \frac{1}{[x^2]^{2\omega-3}}.$$

If we now combine this with the analogous expression for the vector field propagator and compute the correction to the circular loop the result is

$$\Sigma_2 = -g^4 N^2 \frac{\Gamma^2(\omega-1)}{2^7 \pi^{2\omega} (2-\omega)(2\omega-3)} \oint d\tau_1\, d\tau_2 \frac{\left|\dot{x}^{(1)}\right| \left|\dot{x}^{(2)}\right| - \dot{x}^{(1)} \cdot \dot{x}^{(2)}}{\left[(x^{(1)} - x^{(2)})^2\right]^{2\omega-3}}, \tag{12}$$

where a factor of $N^2/2$ came from taking the trace over gauge group generators, a factor of $1/N$ came from the normalization of the Wilson loop, and an additional factor of $1/2$ came from the combinatorics of expanding the Wilson loop operator to second order. We see that the integrand is identical to (2) with a correction to the coefficient

$$\frac{g^2 N}{4\pi^2} \mapsto \frac{g^2 N}{4\pi^2} - \frac{g^4 N^2 \Gamma^2(\omega-1)}{128\,\pi^{2\omega}(2-\omega)(2\omega-3)}.$$

The coefficient diverges in four dimensions (at $\omega = 2$).

2.4 Diagrams with one internal vertex

The order $g^4 N^2$ contributions with one internal vertex come when we Taylor expand $W(C)$ to third order in A and Φ and Wick contract it with the relevant vertices to obtain the quantities:

$$\frac{i^3}{3!}\int d\tau_1\, d\tau_2\, d\tau_3 \Big\langle \mathrm{Tr}\,\mathcal{P}[A(\tau_1)A(\tau_2)A(\tau_3)]\Big(-\int d^4y\, f^{abc}\partial_\mu \phi_i^a(y) A_\mu^b(y)\phi_i^c(y)\Big)\Big\rangle,$$
$$\frac{i}{2!1!}\int d\tau_1\, d\tau_2\, d\tau_3 \Big\langle \mathrm{Tr}\,\mathcal{P}[\Phi(\tau_1)A(\tau_2)\Phi(\tau_3)]\Big(-\int d^4y\, f^{abc}\partial_\mu A_\nu^a(y) A_\mu^b(y) A_\nu^c(y)\Big)\Big\rangle,$$

where $A(\tau) = A_\mu^a(x)\dot{x}^\mu(\tau)T^a$ and $\Phi(\tau) = \Phi^a(x)|\dot{x}|T^a$. The minus signs in both vertices come from the expansion of e^{-S} (we work in Euclidean space). The sum of the two diagrams is

$$\begin{aligned}\Sigma_3 = -\frac{g^4 N^2}{4}&\oint d\tau_1\, d\tau_2\, d\tau_3\, \epsilon(\tau_1\,\tau_2\,\tau_3)(|\dot{x}^{(1)}||\dot{x}^{(3)}| - \dot{x}^{(1)}\cdot\dot{x}^{(3)}) \\ &\times \dot{x}^{(2)}\cdot\frac{\partial}{\partial x^{(1)}}\int d^{2\omega}w\, \Delta(x^{(1)}-w)\Delta(x^{(2)}-w)\Delta(x^{(3)}-w).\end{aligned} \tag{13}$$

Here, ϵ is the antisymmetric path ordering symbol: we define $\epsilon(\tau_1\,\tau_2\,\tau_3) = 1$ for $\tau_1 > \tau_2 > \tau_3$ and let ϵ be antisymmetric under any transposition of τ_i. It is straightforward to introduce Feynman parameters and do the integral over w to obtain

$$\begin{aligned}\Sigma_3 = g^4 N^2 \frac{\Gamma(2\omega-2)}{2^7\pi^{2\omega}}&\int_0^1 d\alpha\, d\beta\, d\gamma\, (\alpha\beta\gamma)^{\omega-2}\delta(1-\alpha-\beta-\gamma)\oint d\tau_1\, d\tau_2\, d\tau_3\, \epsilon(\tau_1\,\tau_2\,\tau_3) \\ \times\, &\frac{\big(|\dot{x}^{(1)}||\dot{x}^{(3)}| - \dot{x}^{(1)}\cdot\dot{x}^{(3)}\big)\big(\alpha(1-\alpha)\dot{x}^{(2)}\cdot x^{(1)} - \alpha\gamma\dot{x}^{(2)}\cdot x^{(3)} - \alpha\beta\dot{x}^{(2)}\cdot x^{(2)}\big)}{\big[\alpha\beta|x^{(1)}-x^{(2)}|^2 + \alpha\gamma|x^{(1)}-x^{(3)}|^2 + \beta\gamma|x^{(3)}-x^{(2)}|^2\big]^{2\omega-2}}.\end{aligned} \tag{14}$$

3 Cancellation of divergences to order g^4N^2

A logarithmic divergence arises in the integral (13) from where τ_1 is coincident with τ_2. This divergence should cancel with the divergence in the coefficient of Σ_2 in (12) so that the order g^4N^2 contribution

$$\Sigma_1 + \Sigma_2 + \Sigma_3$$

is finite. In extracting the divergences from (13), we must consider the integral

$$G(\tau_i) = \int d^4w\, \Delta(w - x^{(1)})\Delta(w - x^{(2)})\Delta(w - x^{(3)}), \tag{15}$$

in detail. This integral is singular in the limit $x^{(2)} \to x^{(1)}$. The divergent contribution comes from the integration over w close to $x^{(1)}$ and $x^{(2)}$. We can approximate $|w - x^{(3)}| \approx |x^{(1)} - x^{(3)}|$ introducing simultaneously an infrared cutoff δ, so that the divergent part of (15) takes the form

$$G(\tau_i) \sim \Delta(x^{(1)} - x^{(3)}) \int^{\delta} d^4w\, \Delta(w - x^{(1)})\Delta(w - x^{(2)}). \tag{16}$$

As follows from dimensional counting, this integral depends only on the dimensionless ratio $|x^{(1)} - x^{(2)}|/\delta$. The limit where $x^{(1)}$ and $x^{(2)}$ are coincident is then equivalent to the limit of infinite δ. Without the infrared cutoff the integral would diverge logarithmically, so up to terms regular in the limit $\delta \to \infty$, we get

$$G(\tau_i) \sim \frac{1}{64\pi^6} \frac{1}{|x^{(1)} - x^{(3)}|^2} \int \frac{d^4w}{w^4} = -\frac{1}{64\pi^4} \frac{\log |x^{(1)} - x^{(2)}|^2/\delta^2}{|x^{(1)} - x^{(3)}|^2}. \tag{17}$$

Since G only goes as a logarithm as points approach each other, (13) receives divergent contributions only from $x^{(1)}$ near $x^{(2)}$; the parts of the integral with $x^{(1)}$ or $x^{(2)}$ near $x^{(3)}$ are finite. This divergence is regularized by cutting off the integral over τ_1. Since the overall divergence is logarithmic, the result is independent of the method of regularization. Writing $\tau = \tau^{(1)} - \tau^{(2)}$ and Taylor expanding $x^{(1)} = x^{(2)} + \dot{x}^{(2)}\tau + \cdots$, we see that the divergent part of (13) is

$$\begin{aligned}
\Sigma_3 &\sim -\frac{g^4N^2}{128\pi^4} \oint d\tau_2 \oint d\tau_3 \frac{|\dot{x}^{(2)}||\dot{x}^{(3)}| - \dot{x}^{(2)} \cdot \dot{x}^{(3)}}{|x^{(2)} - x^{(3)}|^2} \int d\tau\ \mathrm{sign}\, \tau \frac{\dot{x}^{(2)} \cdot (x^{(1)} - x^{(2)})}{|x^{(1)} - x^{(2)}|^2} \\
&= -\frac{g^4N^2}{64\pi^4} \oint d\tau_2 \oint d\tau_3 \frac{|\dot{x}^{(2)}||\dot{x}^{(3)}| - \dot{x}^{(2)} \cdot \dot{x}^{(3)}}{|x^{(2)} - x^{(3)}|^2} \log \epsilon.
\end{aligned}$$

This cancels exactly against (12), for one should replace the pole $1/(2-\omega)$ at $\omega = 2$ by $-2\log\epsilon$.

4 The circular loop

In this section, we consider a circular Wilson loop, whose radius we can assume to be unity. A convenient parameterization of this loop is

$$x(\tau) = (\cos\tau, \sin\tau, 0, 0). \tag{18}$$

4.1 Summing the planar ladder graphs

First, we will see how to sum all planar diagrams which have no internal vertices. These include all ladder and rainbow diagrams. Our strategy is the following: the large N limit with g^2N held fixed is given by planar diagrams. It is well known that each planar graph will contain the same group theoretical factors [15]. We observe that, in fact, each planar diagram without internal vertices gives an identical contribution to the loop expectation value. We choose a fixed ordering of the times and compute a particular, convenient diagram. Then we multiply by the number of diagrams that occur to that order and sum over all orders.

First, consider the $2n$-th order term in the Taylor expansion of the loop

$$\frac{1}{N}\int_0^{2\pi} d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{2n-1}} d\tau_{2n}\, \mathrm{Tr}\big\langle \big(iA(\tau_1)+\Phi(\tau_1)\big)\cdots\big(iA(t_{2n})+\Phi(t_{2n})\big)\big\rangle.$$

Here we have chosen a particular time ordering, which cancels the factor of $1/(2n)!$ which would come from the Taylor expansion of the exponential. We are interested in all Wick contractions which represent planar diagrams. Note that for the circular loop

$$\langle\big(iA^a(\tau_1)+\Phi^a(\tau_1)\big)\big(iA^b(\tau_2)+\Phi^b(\tau_2)\big)\rangle_0 = \frac{g^2\delta^{ab}}{4\pi^2}\frac{|\dot{x}^{(1)}||\dot{x}^{(2)}| - \dot{x}^{(1)}\cdot\dot{x}^{(2)}}{|x^{(1)}-x^{(2)}|^2} = \frac{g^2\delta^{ab}}{8\pi^2}. \tag{19}$$

Thus, the contributions of all (free-field) Wick contractions giving planar diagrams are identical (if we sum over all ways of choosing scalar or gluon lines). The color factor can be computed by repeated application of the identity:

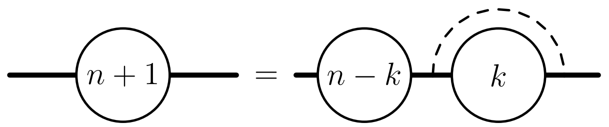
$$T^aT^a = \tfrac{N}{2}\,\mathbf{1}.$$

Thus, for the sum of ladder-like diagrams with n propagators we obtain

$$\frac{(g^2N/4)^n}{(2n)!}\times(\#\text{ of planar graphs with } n \text{ internal lines}), \tag{20}$$

where the factor $1/(2n)!$ accounts for the integral over the $\tau^{(i)}$.

We now have the task of counting the number of planar graphs with n internal lines. Any such diagram with $n+1$ propagators can be uniquely decomposed as



If we define A_{n+1} as the number of such diagrams then A_{n+1} satisfies the recursion relation

$$A_{n+1} = \sum_{k=0}^{n} A_{n-k} A_k,$$

with $A_0 = 1$. If we define a generating function f by

$$f(z) = \sum_{n=0}^{\infty} A_n z^n,$$

then f satisfies $zf^2(z) = f(z) - 1$. So,

$$f(z) = \frac{1 - \sqrt{1-4z}}{2z} = \sum_{n=0}^{\infty} \frac{(2n)!}{(n+1)!n!} z^n. \tag{21}$$

The sign of the square root is chosen by requiring that f be finite at $z = 0$. Hence

$$A_n = \frac{(2n)!}{(n+1)!n!}, \tag{22}$$

so the sum of *all* planar diagrams without vertices on the loop is, from (20) and (22)

$$\langle W(C)\rangle_{\text{ladders}} = \sum_{n=0}^{\infty} \frac{(g^2N/4)^n}{(n+1)!n!} = -\frac{2}{\sqrt{g^2N}} I_1(\sqrt{g^2N}). \tag{23}$$

Thus, the large g^2N behavior is

$$\langle W(C)\rangle_{\text{ladders}} \sim \frac{e^{\sqrt{g^2N}}}{(\pi/2)^{1/2}(g^2N)^{3/4}}. \tag{24}$$

The supergravity prediction is that

$$\langle W(C)\rangle_{\text{AdS/CFT}} \sim e^{\sqrt{g^2N}}, \tag{25}$$

so the ladder diagrams have the same leading behavior as the prediction of the AdS/CFT correspondence.

It is worth mentioning that the cancellation of coordinate dependence in the Wick contraction (19) maps the problem of summing the ladder-like diagrams to the zero dimensional theory. In particular, the number of planar graphs with no vertices and n propagators can be calculated from the infinite N limit of the matrix integral [16]

$$A_n = \left\langle \frac{1}{N} \operatorname{Tr} M^{2n} \right\rangle = \frac{1}{Z} \int [dM] \, \frac{1}{N} \operatorname{Tr} M^{2n} \exp\left\{ -\frac{N}{2} \operatorname{Tr} M^2 \right\},$$

where

$$Z = \int [dM] \, \exp\left\{ -\frac{N}{2} \operatorname{Tr} M^2 \right\}.$$

This can be evaluated by extracting the term of order z^{2n} in the Taylor expansion of the zero-dimensional Wilson loop [17]:

$$\Omega(z) = \left\langle \frac{1}{N} \operatorname{Tr} \frac{1}{1 - zM} \right\rangle. \tag{26}$$

Using the identity

$$\frac{1}{(2n)!} = \oint_{\mathcal{C}} \frac{dz}{2\pi i} \frac{e^z}{z^{2n+1}}$$

for a positively oriented contour $\mathcal{C}$ containing the origin, we can represent the sum of the ladder diagrams as

$$\langle W(C) \rangle_{\text{ladders}} = \oint_{\mathcal{C}} \frac{dz}{2\pi i} \frac{e^z}{z} \Omega(g^2 N/4z), \tag{27}$$

where $\mathcal{C}$ must be chosen large enough to encircle all singularities of the integrand. The function $\Omega(z)$ satisfies a zero-dimensional loop equation [17] that follows from the identity

$$0 = \int [dM] \frac{\partial}{\partial M_{ij}} \left\{ \left(\frac{1}{1 - zM} \right)_{ij} \exp(-\tfrac{N}{2} \operatorname{Tr} M^2) \right\}$$

and large N factorization. In the infinite N limit, this equality reduces to the algebraic equation for $\Omega(z)$:

$$z \Omega^2(z) - \frac{1}{z} \Omega(z) + \frac{1}{z} = 0.$$

This has solution

$$\Omega(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z^2}.$$

Substituting $z^2 \to z$, this is the same as (21). Shrinking the contour of integration in (27) to the branch cut of $\Omega(z)$, we obtain

$$\langle W(C) \rangle_{\text{ladders}} = 4 \int_{-1}^{1} \frac{dx}{2\pi} e^{\sqrt{g^2 N} x} \sqrt{1 - x^2},$$

which is just the Bessel function (23).

4.2 Diagrams with vertices

The sum of the diagrams with one internal vertex attaching to three points on the Wilson loop is given by the expression (14). It is convenient to abbreviate $\tau_{ij} = \tau_i - \tau_j$. For a circular loop, $|x^{(i)}|^2 = 1$, $|x^{(i)} - x^{(j)}|^2 = 2(1 - \cos\tau_{ij})$, $x^{(i)} \cdot \dot{x}^{(j)} = \sin\tau_{ij}$, and $\dot{x}^{(i)} \cdot \dot{x}^{(j)} = \cos\tau_{ij}$. Thus, from (13)

$$\Sigma_3 = g^4 N^2 \frac{\Gamma(2\omega-2)}{2^{2\omega+5}\pi^{2\omega}} \int_0^1 d\alpha\, d\beta\, d\gamma\, (\alpha\beta\gamma)^{\omega-2} \delta(1-\alpha-\beta-\gamma)$$
$$\times \oint d\tau_1\, d\tau_2\, d\tau_3\, \frac{\epsilon(\tau_1\,\tau_2\,\tau_3)(1-\cos\tau_{13})\big(\alpha(1-\alpha)\sin\tau_{12} + \alpha\gamma\sin\tau_{23}\big)}{\big[\alpha\beta(1-\cos\tau_{12}) + \beta\gamma(1-\cos\tau_{23}) + \gamma\alpha(1-\cos\tau_{13})\big]^{2\omega-2}}. \tag{28}$$

We are going to use integration by parts to rewrite (28) as a sum of a term which will cancel with the order g^4N^2 diagrams with internal vertices in (12) and a term which vanishes when $\omega = 2$. For compactness, write the denominator

$$\Delta = \alpha\beta(1-\cos\tau_{12}) + \beta\gamma(1-\cos\tau_{23}) + \gamma\alpha(1-\cos\tau_{13}).$$

Consider the identity

$$\oint d\tau_1\, d\tau_2\, d\tau_3\, \frac{\partial}{\partial\tau_1} \frac{\epsilon(\tau_1\,\tau_2\,\tau_3)(1-\cos\tau_{13})}{\Delta^{2\omega-3}} = 0. \tag{29}$$

Using

$$\frac{\partial}{\partial\tau_1}\epsilon(\tau_1\,\tau_2\,\tau_3) = 2\delta(\tau_{12}) - 2\delta(\tau_{13}), \tag{30}$$

that $\alpha+\beta+\gamma = 1$, and the fact that the integrand in vanishes when $\tau_1 = \tau_3$, we get

$$\oint d\tau_1\, d\tau_2\, d\tau_3 \bigg\{ -\frac{\sin\tau_{13}\big(\alpha\beta(1-\cos\tau_{12}) + \beta\gamma(1-\cos\tau_{23}) + \gamma\alpha(1-\cos\tau_{13})\big)}{\Delta^{2\omega-2}}$$
$$+ (2\omega-3)\frac{(1-\cos\tau_{13})(\alpha\beta\sin\tau_{12} + \gamma\alpha\sin\tau_{13})}{\Delta^{2\omega-2}}\bigg\}\epsilon(\tau_1\,\tau_2\,\tau_3)$$
$$= 2\oint d\tau_1\, d\tau_2\, \frac{1}{[\gamma(1-\gamma)]^{2\omega-3}} \frac{1}{[1-\cos\tau_{12}]^{2\omega-4}}. \tag{31}$$

Now, change variables in the first term on the left hand side so that the cosines in the denominator all have argument τ_{13} (note that this involves permuting α, β, and γ as well, so the following holds only after inserting $\delta(1-\alpha-\beta-\gamma)$ and integrating these parameters over the unit cube). Then add part of the second term, so that the

remaining part is proportional to $(2\omega-4)$ to obtain

$$\oint d\tau_1\, d\tau_2\, d\tau_3\, \epsilon(\tau_1\,\tau_2\,\tau_3)\Big\{\frac{(1-\cos\tau_{13})\big(\alpha(1-\alpha)\sin\tau_{12}+\alpha\gamma\sin\tau_{23}\big)}{\Delta^{2\omega-2}} + (2\omega-4)\frac{(1-\cos\tau_{13})(\alpha\beta\sin\tau_{12}+\gamma\alpha\sin\tau_{13})}{\Delta^{2\omega-2}}\Big\}$$
$$= 2\oint d\tau_1\, d\tau_2\, \frac{1}{[\gamma(1-\gamma)]^{2\omega-3}}\frac{1}{[1-\cos\tau_{12}]^{2\omega-4}}.$$

The first term on the left hand side is precisely the term occurring in (28). Note that the second term can be rewritten as

$$-\frac{2\omega-4}{2\omega-3}\oint d\tau_1\, d\tau_2\, d\tau_3\, \epsilon(\tau_1\,\tau_2\,\tau_3)(1-\cos\tau_{13})\frac{\partial}{\partial\tau_1}\frac{1}{\Delta^{2\omega-3}}$$
$$=\frac{2\omega-4}{2\omega-3}\Big\{\oint d\tau_1\, d\tau_2\, d\tau_3\, \epsilon(\tau_1\,\tau_2\,\tau_3)\frac{\sin\tau_{13}}{\Delta^{2\omega-3}} + 2\oint d\tau_1\, d\tau_2\, \frac{1}{[\gamma(1-\gamma)]^{2\omega-3}}\frac{1}{[1-\cos\tau_{12}]^{2\omega-4}}\Big\},$$

using integration by parts. Finally, we have

$$\oint d\tau_1\, d\tau_2\, d\tau_3\, \epsilon(\tau_1\,\tau_2\,\tau_3)\frac{(1-\cos\tau_{13})\big(\alpha(1-\alpha)\sin\tau_{12}+\alpha\gamma\sin\tau_{23}\big)}{\Delta^{2\omega-2}}$$
$$=-\frac{2\omega-4}{2\omega-3}\oint d\tau_1\, d\tau_2\, d\tau_3\, \epsilon(\tau_1\,\tau_2\,\tau_3)\frac{\sin\tau_{13}}{\Delta^{2\omega-3}} + \frac{2}{2\omega-3}\oint d\tau_1\, d\tau_2\, \frac{1}{[\gamma(1-\gamma)]^{2\omega-3}}\frac{1}{[1-\cos\tau_{12}]^{2\omega-4}}.$$

If we symmeterize, the integral in the first term on the right hand side may be rewritten (setting $\omega=2$) as

$$\frac{1}{3}\frac{2\omega-4}{2\omega-3}\oint d\tau_1\, d\tau_2\, d\tau_3\, \epsilon(\tau_1\,\tau_2\,\tau_3)\frac{\sin\tau_{12}+\sin\tau_{23}+\sin\tau_{31}}{\alpha\beta(1-\cos\tau_{12})+\beta\gamma(1-\cos\tau_{23})+\gamma\alpha(1-\cos\tau_{13})}.$$

This integration is completely finite. Since this term appears with a coefficient that vanishes when $\omega=2$, it vanishes in four dimensions. Inserting this expression into (28), we have the result

$$\Sigma_3 = g^4N^2\frac{\Gamma^2(\omega-1)}{2^{2\omega+4}\pi^{2\omega}(2\omega-3)(2-\omega)}\oint d\tau_1\, d\tau_2\, \frac{1}{[1-\cos\tau_{12}]^{2\omega-4}} + \mathcal{O}(2\omega-4). \quad (32)$$

If we specialize (12) to the case of a circular loop, we obtain

$$\Sigma_2 = -g^4N^2\frac{\Gamma^2(\omega-1)}{2^{2\omega+4}\pi^{2\omega}(2-\omega)(2\omega-3)}\oint d\tau_1\, d\tau_2\, \frac{1}{[1-\cos\tau_{12}]^{2\omega-3}}. \quad (33)$$

The two contributions (32) and (33) cancel exactly when $2\omega=4$:

$$\Sigma_2+\Sigma_3=0.$$

5 Anti-Parallel lines

5.1 Summing Ladder Diagrams

It is possible to sum the planar ladder diagrams for antiparallel lines separated by a distance L [12]. The calculation can be done by considering the more general case of a sum $\Gamma(S,T)$ of ladder diagrams for two finite antiparallel lines of lengths S and T (we take $S,T \gg L$). The result we are interested in arises in studying the large $S=T$ behavior. The sum of ladder diagrams satisfies the recursion relation

$$\Gamma(S,T) = 1 + \int_0^S ds \int_0^T dt \, \Gamma(s,t) \frac{g^2 N}{4\pi^2[(s-t)^2 + L^2]}, \tag{34}$$

where the second factor in the integrand is the sum of vector and scalar propagators. $\Gamma(S,T)$ satisfies the boundary conditions

$$\Gamma(S,0) = \Gamma(0,T) = 1. \tag{35}$$

Taking derivatives of (34) we obtain the differential equation

$$\frac{\partial^2 \Gamma(S,T)}{\partial S \, \partial T} = \frac{g^2 N}{4\pi^2 \left[(S-T)^2 + L^2\right]} \Gamma(S,T). \tag{36}$$

This equation is separable in the variables $x = (S-T)/L$ and $y = (S+T)/L$. (However, as we shall see in the following, the boundary conditions are not conveniently expressed in terms of these variables.) The separated ansatz is

$$\Gamma[x,y] = \sum_n c_n \psi_n(x) \exp(\Omega_n y/2), \tag{37}$$

where c_n are constants which must be determined so that the boundary condition (35) is satisfied and $\psi_n(x)$ are solutions of the Schrödinger equation

$$\left[-\frac{d^2}{dx^2} - \frac{g^2 N}{4\pi^2(x^2+1)} \right] \psi_n(x) = -\frac{\Omega_n^2}{4} \psi_n(x). \tag{38}$$

The largest Ω_n with non-vanishing contribution in (37) dominates the large y for fixed x (large S and T) asymptotics of Γ. In order to see that the lowest bound state of the Schrödinger operator in (38) contributes, we consider the Laplace transform of $\Gamma[x,y]$. Accounting for the boundary condition (35) which implies that $\Gamma[x,y]$ vanishes when $y = |x|$, we see that the Laplace transform is the resolvent of the Schrödinger equation,

$$\int_{|x|}^\infty dy \, e^{-py} \Gamma(x,y) = 2 \sum_n \frac{\bar{\psi}_n(0) \psi_n(x)}{p^2 - \Omega_n^2/4}.$$

Since the potential is symmetric and the ground state has no nodes ($\psi_0(0) \neq 0$), the ground state eigenvalue gives the largest Ω_n contributing to (37).

It is possible to find the ground state eigenvalues of (38) for small and large g^2N. We find

$$\ln \Gamma(T,T) = \left(g^2N/4\pi - \frac{g^4N^2}{8\pi^3} \ln \frac{1}{g^2N} + \cdots \right) \frac{T}{L}, \tag{39}$$

for $g^2N \ll 1$ and

$$\ln \Gamma(T,T) = \left(\sqrt{g^2N}/\pi - 1 + \mathcal{O}(1/\sqrt{g^2N}) \right) \frac{T}{L}, \tag{40}$$

for $g^2N \gg 1$.

5.2 Diagrams with vertices

The sum of the diagrams with one internal vertex and three lines going to the Wilson loop is given by (14). For anti-parallel lines, τ_1 and τ_3 must be on opposite lines so that the factor $\left|\dot{x}^{(1)}\right| \left|\dot{x}^{(3)}\right| - \dot{x}^{(1)} \cdot \dot{x}^{(3)}$ is non-zero. In that case, it provides a factor of 2. Furthermore, we will find a non-vanishing result only when τ_2 is on the same line as τ_1. There are two possible configurations like this, which provides a further factor of 2. Taking the parameterization of the lines to be

$$\begin{aligned} x^{(1)} &= (\tau_1, L/2, 0, 0) \\ x^{(2)} &= (\tau_2, L/2, 0, 0) \\ x^{(3)} &= (-\tau_3, -L/2, 0, 0) \end{aligned}$$

we obtain

$$\begin{aligned} \Sigma_3 = g^4N^2 \frac{\Gamma(2\omega-2)}{2^5\pi^{2\omega}} \int_0^1 d\alpha\, d\beta\, d\gamma\ (\alpha\beta\gamma)^{\omega-2}\, \delta(1-\alpha-\beta-\gamma) \int d\tau_1\, d\tau_2\, d\tau_3\, \epsilon(\tau_1\, \tau_2\, \tau_3) \\ \times \frac{[\alpha\beta(\tau_1-\tau_2) + \gamma\alpha(\tau_1+\tau_3)]}{\left[\alpha\beta(\tau_1-\tau_2)^2 + \beta\gamma\left((\tau_2+\tau_3)^2+L^2\right)^2 + \gamma\alpha\left((\tau_3+\tau_1)^2+L^2\right)^2\right]^{2\omega-2}}, \end{aligned}$$

which can be written as

$$\begin{aligned} \Sigma_3 = -g^4N^2 \frac{\Gamma(2\omega-3)}{2^6\pi^{2\omega}} \int_0^1 d\alpha\, d\beta\, d\gamma\ (\alpha\beta\gamma)^{\omega-2}\, \delta(1-\alpha-\beta-\gamma) \int d\tau_1\, d\tau_2\, d\tau_3\, \epsilon(\tau_1\, \tau_2\, \tau_3) \\ \times \frac{\partial}{\partial\tau_1} \frac{1}{\left[\alpha\beta(\tau_1-\tau_2)^2 + \beta\gamma\left((\tau_2+\tau_3)^2+L^2\right) + \gamma\alpha\left((\tau_3+\tau_1)^2+L^2\right)\right]^{2\omega-3}}. \end{aligned}$$

Then, using (30) we can do the integral over τ_1 to obtain

$$\Sigma_3 = g^4 N^2 \frac{\Gamma(2\omega-3)}{2^5\pi^{2\omega}} \int_0^1 d\alpha\, d\beta\, d\gamma\, \frac{(\alpha\beta\gamma)^{\omega-2}\delta(1-\alpha-\beta-\gamma)}{\left[\gamma(1-\gamma)\right]^{2\omega-3}} \times \int_{-\infty}^{\infty} d\tau_2\, d\tau_3\, \frac{1}{\left[(\tau_3+\tau_2)^2+L^2\right]^{2\omega-3}},$$

which yields

$$\Sigma_3 = g^4 N^2 \frac{\Gamma^2(\omega-1)}{32\pi^{2\omega}(2\omega-3)(2-\omega)} \int_{-\infty}^{\infty} d\tau_2\, d\tau_3\, \frac{1}{\left[(\tau_3+\tau_2)^2+L^2\right]^{2\omega-3}}.$$

This result is to be added to the diagrams which come from exchange of a vector and scalar field, each with internal loop corrections. The sum of those two diagrams is

$$\Sigma_2 = -g^4 N^2 \frac{\Gamma^2(\omega-1)}{32\pi^{2\omega}(2\omega-3)(2-\omega)} \int d\tau_2\, d\tau_3\, \frac{1}{\left[(\tau_3+\tau_2)^2+L^2\right]^{2\omega-3}}, \tag{41}$$

and, in the end,

$$\Sigma_2 + \Sigma_3 = 0.$$

We see that the quantum corrections to the Wilson loop cancel identically in all spacetime dimensions less than ten.

6 Conclusions

We have calculated contribution of planar diagrams without internal vertices to Wilson loops in $\mathcal{N}=4$ supersymmetric Yang-Mills theory. For the two particular types of loops we have considered, circular and rectangular, these diagrams exponentiate. This fact is rather unexpected, given that the combinatorics of planar diagrams without vertices is different from that of diagrams in the free $U(1)$ theory [18, 19, 20], which exponentiate for obvious reasons. Because the ladder-like diagrams exponentiate, the static potential is well defined in the ladder approximation and is easy to extract from the expectation values of the rectangular Wilson loop, (39) and (40) [12]. The dependence of the potential on the coupling constant, when extrapolated to the strong coupling regime, is very similar to, but not exactly the same as the one predicted by the AdS/CFT correspondence. We have no explanation for this similarity, but the result for the circular loop indicates that this is not a mere coincidence.

The planar diagrams without internal vertices for the circular Wilson loop sum up into an expression for which the strong coupling limit coincides with the prediction

of the AdS/CFT correspondence. We have conjectured that diagrams with internal vertices cancel for this Wilson loop in $\mathcal{N}=4$ supersymmetric Yang-Mills theory. This conjecture is supported both by a direct calculation at order g^4N^2 and by agreement with the strong coupling results of the AdS/CFT correspondence. It seems that conformal symmetry plays a key role here, as the conformal symmetry of the theory would naïvely suggest that all higher orders of perturbation theory should cancel. While this is not the case, for reasons discussed above, it may be true that the diagrams with internal vertices are constrained by conformal symmetry to cancel amongst themselves. This is supported by the fact that the remarkable cancellation occurs only in four dimensions, the dimensionality in which conformal symmetry is present.

Acknowledgments

This work is supported in part by NSERC of Canada. G.W.S. is supported by the the Ambrose Monell Foundation. The work of K.Z. is supported by PIMS. G.W.S. thanks Steve Adler, Ken Intriligator, Sasha Polyakov and Frank Wilczek for helpful discussions. J.K.E. and K.Z. thank Yuri Makeenko for helpful discussions.

A Useful formulae and notation

The Euclidean space action of four dimensional $\mathcal{N}=4$ supersymmetric Yang-Mills theory is

$$S=\int d^4x\frac{1}{2g^2}\Big\{\frac{1}{2}\left(F^a_{\mu\nu}\right)^2+\left(\partial_\mu\phi^a_i+f^{abc}A^b_\mu\phi^c_i\right)^2+\bar\psi^a i\gamma^\mu\left(\partial_\mu\psi^a+f^{abc}A^b_\mu\psi^c\right)$$
$$+if^{abc}\bar\psi^a\Gamma^i\phi^b_i\psi^c-\sum_{i<j}f^{abc}f^{ade}\phi^b_i\phi^c_j\phi^d_i\phi^e_j+\partial_\mu\bar c^a\left(\partial_\mu c^a+f^{abc}A^b_\mu c^c\right)+\xi(\partial_\mu A^a_\mu)^2\Big\} \tag{42}$$

where

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+f^{abc}A^b_\mu A^c_\nu$$

and f^{abc} are the structure constants of the $U(N)$ Lie algebra,

$$\left[T^a,T^b\right]=if^{abc}T^c.$$

The generators are normalized as

$$\mathrm{Tr}\,T^aT^b=\frac{1}{2}\delta^{ab},$$

and obey the identity

$$\sum_{c,d} f^{acd} f^{bcd} = N\delta^{ab}.$$

Now, ψ^a is a sixteen component spinor obeying the Majorana condition

$$\psi(x) = C\psi^*(x), \tag{43}$$

where C is the charge conjugation matrix. $\Gamma^A = (\gamma^\mu, \Gamma^i)$, for $\mu = 1, ..., 4$ and $i = 5, ..., 10$ are ten real 16×16 Dirac matrices (in the 10-dimensional Majorana representation with the Weyl constraint) obeying

$$\mathrm{Tr}\left(\Gamma^A\Gamma^B\right) = 16\delta^{AB}.$$

We have chosen the covariant gauge fixing condition,

$$\partial_\mu A^a_\mu = 0, \tag{44}$$

and we work in Feynman gauge, where the gauge parameter is chosen as $\xi = 1$. The appropriate action for ghost fields, $c^a(x)$ has been included.

In Feynman gauge, the vector field propagator is

$$\Delta^{ab}_{\mu\nu}(p) = \sim\!\sim\!\sim = g^2\delta^{ab}\frac{\delta_{\mu\nu}}{p^2},$$

the scalar propagator is

$$D^{ab}_{ij}(p) = \rule{3em}{0.4pt} = g^2\delta^{ab}\frac{\delta_{ij}}{p^2},$$

the fermion propagator is

$$S^{ab}(p) = \text{- - - -} = g^2\delta^{ab}\frac{-\gamma\cdot p}{p^2},$$

and the ghost propagator is

$$C^{ab}(p) = \cdots\cdots = g^2\delta^{ab}\frac{1}{p^2}.$$

The vertices can be easily deduced from the non-quadratic terms in the action (42). Each vertex carries a factor of $1/g^2$ and each propagator carries a factor of g^2.

We use the position-space propagators in 2ω-dimensions. These can be deduced from the Fourier transform

$$\int \frac{d^{2\omega}p}{(2\pi)^{2\omega}}\frac{e^{ip\cdot x}}{[p^2]^s} = \frac{\Gamma(\omega - s)}{4^s\pi^\omega\Gamma(s)}\frac{1}{[x^2]^{\omega-s}}. \tag{45}$$

By setting $s=1$ we find the Green function in 2ω dimensions:

$$\Delta(x) = \frac{\Gamma(\omega-1)}{4\pi^\omega}\frac{1}{[x^2]^{\omega-1}} \qquad \text{which satisfies} \qquad -\partial^2\Delta(x) = \delta^{2\omega}(x). \qquad (46)$$

We record some formulae which are useful in reproducing the computations in this paper. Euler's B function is defined as

$$B(\mu,\nu) = \int_0^1 dx\, x^{\mu-1}(1-x)^{\nu-1} = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$$

and the Γ function is

$$\Gamma(n+1) = \int_0^\infty dt\, t^n e^{-t},$$

which satisfies $\Gamma(n+1)=n!$, $\Gamma(1/2)=\sqrt{\pi}$, and the combinatorial formulae

$$\Gamma\big((2n+1)/2\big) = (n-1/2)(n-3/2)\dots(1/2)\sqrt{\pi},$$
$$\Gamma(n)\Gamma(1/2) = 2^{n-1}\Gamma(n/2)\Gamma\big((n+1)/2\big).$$

One loop integrals in 2ω dimensions can be computed using the dimensional regularization formulae [14]:

$$\int d^{2\omega}k(k^2+2p\cdot k+m^2)^{-s} = \pi^\omega\frac{\Gamma(s-\omega)}{\gamma(s)}\left(m^2-p^2\right)^{\omega-s}$$
$$\int d^{2\omega}kk_\mu(k^2+2p\cdot k+m^2)^{-s} = -p_\mu\pi^\omega\frac{\Gamma(s-\omega)}{\gamma(s)}\left(m^2-p^2\right)^{\omega-s}$$
$$\int d^{2\omega}k\, k_\mu k_\nu(k^2+2p\cdot k+m^2)^{-s} = \pi^\omega\frac{1}{\Gamma(s)}(m^2-p^2)^{\omega-s}$$
$$\times\left[p_\mu p_\nu\Gamma(s-\omega) - \tfrac{1}{2}g_{\mu\nu}\Gamma(s-\omega-1)(p^2+m^2)\right]$$

and the Feynman parameter formula,

$$\prod_i A_i^{-n_i} = \frac{\Gamma(\sum n_i)}{\prod_i\Gamma(n_i)}\int_0^1 dx_1\cdots dx_k\, x_1^{n_1-1}\cdots x_k^{n_k-1}\frac{\delta(1-\sum x_i)}{\left[\sum_i A_i x_i\right]^{\sum n_i}}.$$

References

[1] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998), `hep-th/9711200`.

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Phys. Lett. **B428**, 105 (1998), `hep-th/9802109`.

[3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998), hep-th/9802150.

[4] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, hep-th/9905111, and references therein.

[5] For other reviews, see E.T. Akhmedov, hep-th/9911095; P. DiVecchia, Lectures given at ICTP Trieste Spring Workshop on Superstrings and Related Matters, Trieste, Italy, 22-30 Mar 1999, hep-th/9908148; J.L. Petersen, Int.J.Mod.Phys.A14:3597-3672,1999 hep-th/9902131.

[6] J. Maldacena, Phys. Rev. Lett. **80**, 4859 (1998), hep-th/9803002.

[7] S. Rey and J. Yee, hep-th/9803001.

[8] N. Drukker, D.J. Gross and H. Ooguri, Phys.Rev.D60:125006,1999, hep-th/9904191.

[9] J. Sonnenschein, hep-th/0003032.

[10] W. Chen, Y. Makeenko and G.W. Semenoff, Annals Phys.228:341-364,1993, hep-th/9301069.

[11] D. Berenstein, R. Corrado, W. Fischler and J. Maldacena, Phys. Rev. **D59**, 105023 (1999), hep-th/9809188.

[12] J.K. Erickson, G.W. Semenoff, R.J. Szabo and K. Zarembo, hep-th/9911088, to appear in Phys. Rev. D.

[13] T. Appelquist, M. Dine and I.J. Muzinich, Phys. Rev. D17, 2074 (1978).

[14] P. Ramond, *Field Theory: A Modern Primer*, Addison-Wesley, 1989.

[15] G. 't Hooft, Comm. Math. Phys. **86**, 449 (1982).

[16] E. Brezin, C. Itzykson, G. Parisi and J.B. Zuber. Commun. Math. Phys. 59, 35 (1978).

[17] Y. Makeenko, Mod. Phys. Lett. **A6**, 1901 (1991) and references therein.

[18] P. Cvitanovic, Phys. Lett. **B99**, 49 (1981).

[19] P. Cvitanovic, P. G. Lauwers and P. N. Scharbach, Nucl. Phys. **B203**, 385 (1982).

[20] R. Gopakumar and D. J. Gross, Nucl. Phys. **B451**, 379 (1995), hep-th/9411021.